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THE ELECTROMAGNETIC FIELD PRODUCED BY A HELIX

by

RALPH S. PHILLIPS

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Abstract

The electromagnetic field is found for an infinitely-thin perfectly-conducting helix subject to a monochromatic excitation. It is first assumed that such a field would induce a sinusoidal current on the helix. The retarded vector potential is then constructed from this assumed current. Finally the propagation constant for this current is determined so that the electric field along the helix is 'essentially' zero, thus establishing the correctness of the initial assumption. The phase velocity along the helix itself which corresponds to this propagation constant is in fact equal to the free space velocity of propagation. A Bessel function expansion is found for the electromagnetic field from integral evaluations of the type

$$\int_{-\infty}^{\infty} e^{i\beta\zeta} \frac{e^{ikR}}{R} \, d\zeta = 4\pi \sum_{n=-\infty}^{\infty} e^{in\varphi} I_n(\Gamma_n a) K_n(\Gamma_n \rho)$$

where $R = \sqrt{\xi^2 + a^2 + \rho^2 - 2a\rho\cos(\varphi - \zeta/\alpha)}$, $\Gamma_n = \sqrt{(\beta - n/\alpha)^2 - k^2}$, and $\rho > a$.

It turns out that for a given helix there are certain frequency bands in which the system radiates. This result was not present in the cruder helical idealizations previously investigated. A method is suggested whereby one could extend the results of this paper to helices of finite wire diameter in the non-radiating case. The paper concludes with a discussion of energy flow considerations.

1. Introduction

In this paper we are concerned with finding the electromagnetic field produced by an infinitely-thin perfectly-conducting helix subject to a monochromatic excitation. The helix has come in for a considerable amount of attention in recent years because of its use in an ultra-high frequency amplifier known as the traveling wave tube [2] . This tube makes use of the fact that an electromagnetic field travels along the axis of the helix with a phase velocity considerably less than the velocity of light. An electromagnetic signal is introduced at one end of the helix together with an electron beam which is shot down the axis of the helix at about the phase velocity of the signal. The signal and electron beam interact in such a way as to amplify the signal.

The electromagnetic field produced by a helix has been investigated by F. Ollendorff [1] , by S. A. Schelkunoff [2], by R. S. Phillips and H. Malin [3], and others. These investigations were based on a model consisting of a circular cylinder which conducts only in a given helical direction*. The symmetry of this idealization corresponds to that of cylindrical coordinates so that such a model yields to the usual separation of variables approach. We have followed a different line of attack.

We start off by assuming that the effect of the monochromatic electromagnetic field would be to induce a sinusoidal electric current on the helical conductor; and that this current travels along the helix with a real propagation constant β in the axial direction.** The electromagnetic field (for arbitrary β) is in turn represented by means of the retarded vector potential [4, p. 165] . If the initial assumption as to the nature of the induced electric current on the helix is correct, then it is possible to determine β so that the proper boundary conditions at the helix are satisfied. In other words, we attempt to determine β so that the electric field (as determined by the retarded vector potential) vanishes in the helical direction on

---

* By helical direction we mean the direction of a vector field which, in cylindrical coordinates, has the components

$$v_\rho = 0, \quad v_\varphi = a, \quad v_z = \alpha$$

** That is the amplitude varies as $\cos(\beta z - \omega t)$ where z is measured in the axial direction and ω is the radial frequency.

the helix. Due to the singular nature of an infinitely-thin perfect conductor, it is not possible to chose β so that the electric field in the helical direction E_h actually vanishes. It is, however, possible to find a real number β_0 so that the ratio $E_h(\beta_0)/E_h(\beta \neq \beta_0)$ approaches zero as the helix is approached radially. It will turn out that the β_0 so determined corresponds to a phase velocity along the helix itself equal to the free space wave velocity c. In fact

$$\beta_0 = \pm k \frac{\sqrt{a^2+\alpha^2}}{\alpha}$$

where $k^2 = \epsilon\mu\omega^2$, ω being the radial frequency of the field

a = radius of the helix

α = 2π times the distance between turns on the helix.

The retarded vector potential Π is defined as [4,p.165]

$$\Pi(x,y,z,t) = A\int \frac{[i]}{r}\, ds \, , \tag{1.1}$$

where

$[i]$ = current vector at a point (ξ, η, ζ) of the helix at time $(t - r/c)$, c being the free space wave velocity

$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$

s = arc length along the helix

A = scalar factor.

Then

$$\begin{aligned} E &= \nabla(\nabla\cdot\Pi) + k^2\Pi \\ H &= -i\omega\epsilon\,\nabla\times\Pi. \end{aligned} \tag{1.2}$$

Thus once β_0 is determined, Π is known and hence by (1.2) both E and H are known.

We have further been able to evaluate the integral expressions for the electric and magnetic fields given by (1.2) in terms of Bessel function expansions. These are given explicitly in the equations (3.6) and (6.3). One rather surprising result becomes evident on examining these equations. If for some

value of n, we have

$$k^2 > (\beta_0 - n/\alpha)^2 ,$$

then Hankel function terms appear in these expansions. Physically, this means that the system will radiate if the above inequality is satisfied for some n. This result is unexpected on two counts. In the first place in the earlier idealizations of the helix as a helical cylinder, radiating modes were not possible. In the second place radiating terms in the presence of a real propagation constant appears contradictory. However, this can be explained by the fact that the axial flow of energy is always infinite whereas the radiating terms cause only a finite radial flow of energy per unit distance in the axial direction.

In proceeding with this general program, we show in section 2 that the integral defining the retarded vector potential Π converges, a fact which because of the singular nature of the integral is by no means evident. In addition we obtain an evaluation for Π in terms of Bessel functions. The central result here is

$$(1.3) \int_{-\infty}^{\infty} e^{i\beta\zeta} \frac{e^{ikR}}{R} \, d\zeta = \sum_{n=-\infty}^{\infty} 4\pi e^{in\varphi} \, I_n(\Gamma_n a) \, K_n(\Gamma_n \rho) \quad \text{for } \rho > a$$

where

$$R = \sqrt{\zeta^2 + a^2 + \rho^2 - 2a\rho\cos(\varphi - \zeta/\alpha)}$$

and

$$\Gamma_n = \sqrt{(\beta - n/\alpha)^2 - k^2}$$

Here I_n and K_n are modified Bessel and Hankel functions of n^{th} order. If for a particular n, Γ_n is imaginary, then the corresponding term in (1.3) is to be replaced by $2\pi^2 \, i e^{in\varphi} \, J_n(\gamma_n a) \, H_n^{(1)}(\gamma_n \rho)$ where $\gamma_n = \sqrt{k^2 - (\beta - n/\alpha)^2}$

In section 3 we prove that Π may be differentiated either by differentiating under the integral sign or by differentiating its series expansion term-wise. We then show that Π satisfies the homogeneous wave equation at points not on the helix. Also in section 3 explicit expressions for the electromagnetic field are obtained. Finally in section 4 the propagation constant β is determined

so that the boundary conditions for the helix are satisfied.

A concluding section summarizes the findings of the paper and goes more deeply into the physical significance and reasonableness of our result. A method is suggested for extending our results in the absence of radiating terms to the case of a helix made from a finite wire conductor. The appendix is concerned with the details of energy flow calculations.

In cylindrical coordinates, the equation for the helix is

$$\left\{ \begin{array}{l} \rho = a \\ z = \alpha \varphi \end{array} \right. \tag{1.4}$$

The assumed current along the helix is the real part of

$$\begin{aligned} i_\rho &= 0 \\ i_\varphi &= I\, e^{i\beta z} \frac{a}{\sqrt{a^2+\alpha^2}}\, e^{-i\omega t} \\ i_z &= I\, e^{i\beta z} \frac{\alpha}{\sqrt{a^2+\alpha^2}}\, e^{-i\omega t} \end{aligned} \tag{1.5}$$

The retarded potential at the point (ρ, φ, z) and time t is the real part of

$$\begin{aligned} \Pi_\rho &= aF \int_{-\infty}^{\infty} \frac{e^{i\beta\zeta + ikr}}{r} \sin(\varphi - \zeta/\alpha)\, d\zeta \\ \Pi_\varphi &= aF \int_{-\infty}^{\infty} \frac{e^{i\beta\zeta + ikr}}{r} \cos(\varphi - \zeta/\alpha)\, d\zeta \\ \Pi_z &= \alpha F \int_{-\infty}^{\infty} \frac{e^{i\beta\zeta + ikr}}{r}\, d\zeta \end{aligned} \tag{1.6}$$

If we replace sines and cosines by their exponential representations, we see that all of the integrals in (1.6) are essentially of the type found in Π_z. The notation of the next section will be somewhat simplified if we make a change of variable $\zeta' = \zeta - z$ in these integrals. Then

$$\int_{-\infty}^{\infty} e^{i\beta\zeta} \frac{e^{ikr}}{r}\, d\zeta = e^{i\beta z} \int_{-\infty}^{\infty} e^{i\beta\zeta'} \frac{e^{ikr'}}{r'}\, d\zeta' \tag{1.7}$$

where

$$r' = \sqrt{\zeta'^2 + a^2 + \rho^2 - 2a\rho \cos\left[(\varphi - z/\alpha) - \zeta'/\alpha\right]}$$

2. Evaluation of the Integral.

In this section we shall evaluate the integral

$$I = \int_{-\infty}^{\infty} e^{i\beta\zeta} \frac{e^{ikR}}{R} d\zeta \tag{2.1}$$

where

$$R = \sqrt{\zeta^2 + a^2 + \rho^2 - 2a\rho \cos(\varphi - \zeta/\alpha)} \quad .$$

We restrict our considerations to real β since the integral will clearly not converge for complex β.

The usual method of handling improper integrals, namely to transform the path of integration, is not suitable here because R has an infinite set of branch points; these are the solutions of the transcendental equation $R^2=0$. The device which we shall use in evaluating (2.1) consists of replacing the integral along the helix by a double integral over the enveloping cylinder of the helix; this leads to a Bessel function solution. Heuristically we could accomplish this by means of a delta-function $\delta(t)$ of period 2π having the properties

$$\delta(t) = 0 \qquad t \neq 2\pi n \quad (n=0, \pm 1, \pm 2, \dots)$$

$$\int_{-\epsilon}^{\epsilon} \delta(t)dt = 1 \quad \text{for } 0 < \epsilon < 2\pi$$

The integral then takes the form

$$\int_{-\infty}^{\infty} e^{i\beta\zeta} \frac{e^{ikR}}{R} d\zeta = \int_{-\infty}^{\infty} d\zeta \int_0^{2\pi} d\sigma \; e^{i\beta\zeta} \frac{e^{ik\mathcal{R}}}{\mathcal{R}} \; \delta(\sigma - \zeta/\alpha),$$

where in the right hand integral the quantity R has been replaced wherever it occurs by

$$\mathcal{R} = \sqrt{\zeta^2 + a^2 + \rho^2 - 2a\rho \cos(\varphi - \sigma)}. \tag{2.2}$$

We now expand $\delta(t)$ in its Fourier series

$$\delta(t) = \sum_{n=-\infty}^{\infty} e^{int}$$

interchange order of summation and integration to obtain

$$I = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\zeta \int_0^{2\pi} d\sigma \; e^{i(\beta - n/\alpha)\zeta} \frac{e^{ik\mathcal{R}}}{\mathcal{R}} \; e^{in\sigma}$$

An interchange of the order of integration then permits us to evaluate the integral by known Bessel function formulae.

We shall achieve this same end by replacing the delta-function by the better behaved Poisson kernel [6,p.51]. We define

$$I(L,r) = \int_{-L}^{L} d\zeta \int_{0}^{2\pi} d\sigma \; e^{i\beta\zeta} \; \frac{e^{ik\mathcal{R}}}{\mathcal{R}} \; P(r, \sigma - \zeta/\alpha) \tag{2.3}$$

where $P(r, \sigma)$ is the Poisson kernel

$$P(r,\sigma) = \frac{1}{2\pi} \; \frac{1 - r^2}{1 + r^2 - 2r \cos\sigma} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} \; e^{in\sigma}$$

For $0 \leq r < 1$, $L < \infty$, and $\rho > a$ the double integral is absolutely integrable and hence the order of integration may be interchanged. By well known properties of the Poisson kernel

$$\lim_{r \to 1^-} \int_{0}^{2\pi} d\sigma \; e^{i\beta\zeta} \; \frac{e^{ik\mathcal{R}}}{\mathcal{R}} \; P(r, \sigma - \zeta/\alpha) = e^{i\beta\zeta} \; \frac{e^{ikR}}{R}$$

uniformly for $-L \leq \zeta \leq L$. Hence

$$I = \lim_{L \to \infty} \; \lim_{r \to 1^-} I(L,r) \tag{2.4}$$

if this limit exists.

We wish to show not only that the limit of (2.4) exists but that the order of this double limit may be interchanged. In order to accomplish this we shall prove

a) $\lim\limits_{L \to \infty} I(L,r)$ exists for each $r (0 \leq r < 1)$

b) $\lim\limits_{r \to 1^-} I(L,r)$ exists uniformly in L for $(0 < L < \infty)$.

Since the Fourier expansion of $P(r, \sigma)$ converges uniformly in σ for each $r < 1$, it follows that we can interchange the order of integration and

summation and so express I(L,r) as

$$(2.5) \qquad I(L,r) = \sum_{n=-\infty}^{\infty} r^{|n|} \int_{-L}^{L} d\rho \int_{0}^{2\pi} d\sigma \; \frac{e^{i\beta\zeta + ik\mathcal{R}}}{\mathcal{R}} \; e^{in(\sigma - \zeta/\alpha)}$$

In order to establish (a) we first examine the individual terms of this sum. For the case $(\beta - n/\alpha) > k$, we make the substitution

$$\zeta = \sqrt{a^2 + \rho^2 - 2a\rho \cos(\varphi - \sigma)} \quad \sinh(\psi - \delta)$$

$$\mathcal{R} = \sqrt{a^2 + \rho^2 - 2a\rho \cos(\varphi - \sigma)} \quad \cosh(\psi - \delta)$$

where

$$\cosh \delta = \frac{\beta - n/\alpha}{\sqrt{(\beta - n/\alpha)^2 - k^2}}$$

$$\sinh \delta = \frac{k}{\sqrt{(\beta - n/\alpha)^2 - k^2}} \quad .$$

Then

$$(\beta - n/\alpha)\zeta + k\mathcal{R} = \Phi \sinh \psi ,$$

where we have set

$$(2.6) \qquad \Phi = \sqrt{a^2 + \rho^2 - 2a\rho \cos(\varphi - \sigma)} \sqrt{(\beta - n/\alpha)^2 - k^2} \quad .$$

Hence

$$\int_{-L}^{L} d\zeta \;\; e^{i(\beta - n/\alpha)\zeta} \, \frac{e^{ik\mathcal{R}}}{\mathcal{R}} = \int_{-\ell_1}^{\ell_2} d\psi \; e^{i\Phi \sinh \psi} ,$$

where ℓ_1 and ℓ_2 are defined by

$$L = \sqrt{a^2 + \rho^2 - 2a\rho \cos(\varphi - \sigma)} \quad \sinh(\ell_1 + \delta)$$

$$L = \sqrt{a^2 + \rho^2 - 2a\rho \cos(\varphi - \sigma)} \quad \sinh(\ell_2 - \delta) \quad ,$$

δ being defined as above. We now transform the path of this integral to the line $\psi = i\pi/2$. Then because of the fact that the integral along the line joining $(\ell, 0)$ and $(\ell, i\pi/2)$ goes to zero as $\ell \to \pm\infty$, we get

$$(2.7)\quad \int_{-\infty}^{\infty} d\zeta \; e^{i(\beta - n/\alpha)\zeta} \; \frac{e^{ik\mathcal{R}}}{\mathcal{R}} = \int_{-\infty}^{\infty} d\psi \; e^{-\Phi \cosh \psi}$$
$$= 2\, K_0(\Phi)$$

The last equality can be found in [5, p.182, (7)].

It is clear from the above argument that

$$2K_0(\Phi) - \int_{-L}^{L} d\zeta \quad e^{i(\beta - n/\alpha)\zeta} \; \frac{e^{ik\mathcal{R}}}{\mathcal{R}} = \int_P d\psi \; e^{i\Phi \sinh \psi}$$

where P is the sum of the two paths indicated on the accompanying figure.

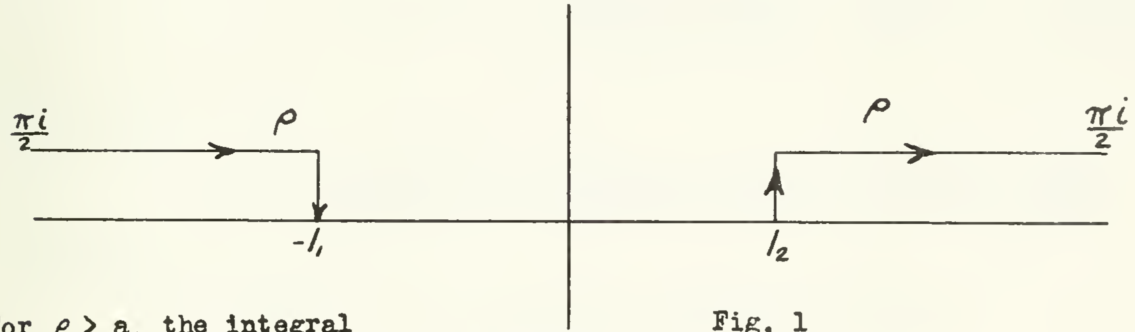


Fig. 1

For $\rho > a$, the integral

$$\int_P d\psi \; e^{i\Phi \sinh \psi}$$

converges to zero uniformly in σ as $\ell_1, \ell_2 \to \infty$. This incidentally shows that the order of integration in

$$\int_0^{2\pi} d\sigma \int_{-\infty}^{\infty} d\zeta \; e^{i\beta\zeta} \; \frac{e^{ik\mathcal{R}}}{\mathcal{R}} \; e^{in(\sigma - \zeta/\alpha)}$$

may be interchanged. We can, however, easily obtain an upper bound in the value of this integral over P. In fact the integral from $(\ell, 0)$ to $(\ell, \pi i/2)$ is bounded by $\dfrac{2}{\Phi \cosh \ell}$; whereas the integral from $(\ell, \pi i/2)$ to $(\infty, \pi i/2)$ is bounded by $\dfrac{e^{-\Phi \ell}}{\Phi}$. Now for sufficiently large n, fixed β, and fixed $\rho > a$,

$\Phi \geq c\ |n|$. Hence in this case

$$(2.8)\quad \left| \int_0^{2\pi} d\ K_o(\ \Phi\ ) e^{in\sigma} - \int_{-L}^{L} d\rho \int_0^{2\pi} d\sigma \quad e^{i\beta\zeta} \ \frac{e^{ik\mathcal{R}}}{\mathcal{R}} \ e^{in(\sigma - \zeta/\alpha)} \right| < \frac{\bar{c}}{|n|} \ .$$

A similar argument shows that (2.7) and (2.8) hold equally well for $(\beta - n/\alpha) < -k$.

For the case $(\beta - n/\alpha)^2 < k^2$, let

$$\zeta = \sqrt{a^2 + \rho^2 - 2a\rho \cos\ (\varphi - \sigma)}\ \sinh\ (\psi - \delta')$$

$$\mathcal{R} = \sqrt{a^2 + \rho^2 - 2a\rho \cos\ (\varphi - \sigma)}\ \cosh\ (\psi - \delta')$$

where now

$$\sinh \delta' = \frac{\beta - n/\alpha}{\sqrt{k^2 - (\beta - n/\alpha)^2}} \qquad \cosh \delta' = \frac{k}{\sqrt{k^2 - (\beta - n/\alpha)^2}}$$

Then for

$$\Phi' = \sqrt{a^2 + \rho^2 - 2a\rho \cos\ (\varphi - \sigma)}\ \sqrt{k^2 - (\beta - n/\alpha)^2}$$

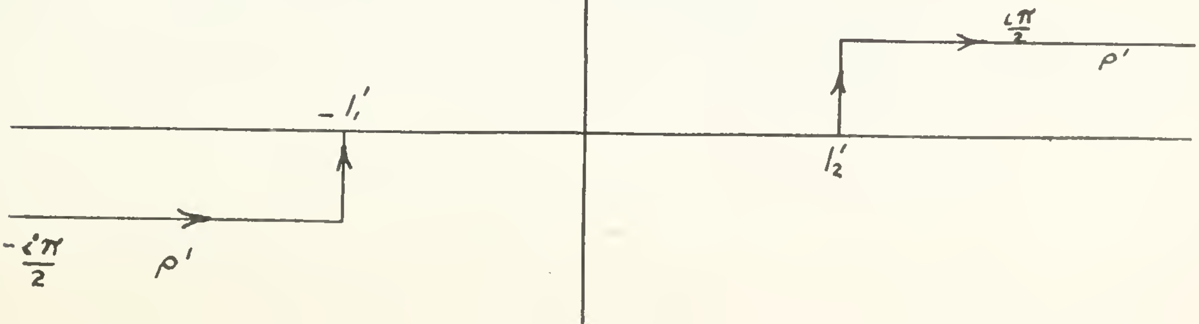
we have

$$i(\beta - n/\alpha)\zeta + ik\mathcal{R} = i\Phi' \cosh \psi ,$$

so that

$$\int_{-L}^{L} d\zeta \quad e^{i(\beta - n/\alpha\ )} \ \frac{e^{ik\mathcal{R}}}{\mathcal{R}} = \int_{-\ell_1'}^{\ell_2'} d\psi \ e^{i\Phi' \cosh \psi} \ .$$

The difference between this integral and the integral from ($-\infty - i\pi/2$) to ($-i\pi/2$) to ($i\pi/2$) to ($\infty + i\pi/2$) is the integral over the path P' shown in the accompanying diagram.



For $\rho > a$, the integral

$$\int_{P'} d\psi \; e^{i\Phi' \cosh\psi}$$

converges to zero uniformly in σ as $\ell_1', \ell_2' \to \infty$. Hence

$$(2.9) \quad \int_{-\infty}^{\infty} d\zeta \quad e^{i(\beta - n/\alpha)\zeta} \; \frac{e^{ik\mathcal{R}}}{\mathcal{R}} = \int_{-\infty - i\pi/2}^{\infty + i\pi/2} d\psi \; e^{i\Phi' \cosh\psi}$$

$$= \pi i \, H_o^{(1)} (\Phi')$$

(see [5, p. 180, (8)]) and

$$(2.10) \quad \int_{-\infty}^{\infty} d\zeta \int_0^{2\pi} d\sigma \, e^{i\beta\zeta} \; \frac{e^{ik\mathcal{R}}}{\mathcal{R}} \, e^{in(\sigma - \zeta/\alpha)} = \int_0^{2\pi} d\sigma \, H_o^{(1)}(\Phi') \, e^{in\sigma} \quad .$$

Now by the addition theorem for Bessel functions [5, p. 361] for $\rho > a$

$$H_o^{(1)}\left(\chi\sqrt{a^2 + \rho^2 - 2a\rho \; \cos\theta}\right) = \sum_{m=-\infty}^{\infty} J_m(xa) \, H_m^{(1)} (x\rho) \, e^{im\theta}$$

$$K_o \; \left(\chi\sqrt{a^2 + \rho^2 - 2a\rho \; \cos\theta}\right) = \sum_{m=-\infty}^{\infty} I_m(xa) \, K_m(x\rho) \, e^{im\theta}$$

and these series converge uniformly in θ.

Substituting these series in the above integrals we obtain for $\rho > a$

$$(2.11) \qquad \frac{1}{2\pi} \int_{-\infty}^{\infty} d\zeta \int_0^{2\pi} d\sigma \; e^{i\beta\zeta} \; \frac{e^{ik\mathcal{R}}}{\mathcal{R}} \, e^{in\,(\sigma - \zeta/\alpha)}$$

$$= 2\, I_n\left(\sqrt{(\beta - n/\alpha)^2 - k^2}\; a\right) K_n\left(\sqrt{(\beta - n/\alpha)^2 - k^2}\; \rho\right) e^{in\varphi} \quad \text{if } (\beta - n/\alpha)^2 > k^2$$

$$= \pi i \; J_n\left(\sqrt{k^2 - (\beta - n/\alpha)^2}\; a\right) H_n^{(1)}\left(\sqrt{k^2 - (\beta - n/\alpha)^2}\; \rho\right) e^{in\varphi} \quad \text{if } (\beta - n/\alpha)^2 < k^2 \, .$$

Now it can be shown [3, p.46] that $I_n(x)\,K_n(x)$ is a monotonic decreasing function of x which assumes the value $\frac{1}{2n}$ at $x = 0$. Since, in addition, $K_n(x)$ is monotonic decreasing in x, it follows that

$$I_n(\Gamma a)\,K_n(\Gamma \rho) < \frac{1}{2n}$$

for $\rho > a$. Combining this with the above result and the inequality (2.8), we obtain for fixed β and fixed $\rho > a$.

$$(2.12)\quad \left| \int_{-L}^{L} d\zeta \int_{0}^{2\pi} d\sigma\; e^{i\beta\zeta}\; \frac{e^{ik\mathcal{R}}}{\mathcal{R}}\; e^{in(\sigma - \zeta/\alpha)} \right| < \frac{c}{n}$$

for sufficiently large n. It follows that $\lim\limits_{L\to\infty} I(L,r)$ exists for $r < 1$ and $\rho > a$, in fact

$$(2.13)\qquad \lim_{L\to\infty} I(L,r) = \sum_{n=-\infty}^{\infty} r^{|n|}\, 4\pi\, I_n(\Gamma_n a)\, K_n(\Gamma_n \rho)\, e^{in\varphi} \quad .$$

If for a particular n, Γ_n is imaginary, then the corresponding term in (2.13) is to be replaced by $2\pi^2 i\, r^{|n|} e^{in\varphi}\; J_n(\gamma_n a)\, H_n^{(1)}(\gamma_n \rho)$ where $\gamma_n = \sqrt{k^2 - (\beta - n/\alpha)^2}$. We shall in the future write the series as above and leave to the reader the interpretation for imaginary Γ_n.

In order to evaluate $\lim\limits_{r\to 1^-} \lim\limits_{L\to\infty} I(L,r)$ we refer back to an earlier paper on the helical wave guide [3, p. 46]. It was there shown that for $n > 0$

$$(2.14)\quad \frac{n}{v^2}\sqrt{1 + \frac{v^2}{n^2}} < -\frac{1}{v}\,\frac{K_n'(v)}{K_n(v)} < \frac{n}{v^2}\sqrt{1 + \frac{v^2}{n(n-1)}}$$

Hence

$$\frac{K_n'(V)}{K_n(v)} < -\frac{n}{v}\sqrt{1 + \frac{v^2}{n^2}} < -\frac{n}{v}$$

and

$$K_n(v) < K_n(v_0)\left(\frac{v_0}{v}\right)^n \qquad \text{for } (v > v_0)\,.$$

This coupled with the earlier mentioned fact that

$$I_n(v)\,K_n(v) \leq \frac{1}{2n}$$

gives

$$I_n(\Gamma_n a)\, K_n(\Gamma_n \rho) \leq \frac{1}{2n} \left(\frac{a}{\rho}\right)^n \tag{2.15}$$

Since $I_{-n}(\Gamma_n a)\, K_{-n}(\Gamma_n \rho) = I_n(\Gamma_n a)\, K_n(\Gamma_n \rho)$, the series 2.13) is majorized uniformly in $0 \leq r \leq 1$ by

$$\sum_{n=-\infty}^{\infty} \frac{1}{2n} \left(\frac{a}{\rho}\right)^{|n|}$$

Thus for $\rho > a$,

$$\lim_{r \to 1^-} \lim_{L \to \infty} I(L,r) = \sum_{n=-\infty}^{\infty} 4\pi\, e^{in\varphi}\, I_n(\Gamma_n a)\, K_n(\Gamma_n \rho) \ . \tag{2.16}$$

Finally we shall establish (b), namely, that the $\lim\limits_{r \to 1^-} I(L,r)$ exists uniformly in L. To this end we define

$$\begin{aligned} h(r,\sigma) &= P(r,\sigma) && \text{for } P(r,\sigma) \leq \epsilon \\ &= \epsilon && \text{for } P(r,\sigma) > \epsilon \end{aligned}$$

$$a(r,\epsilon) = 1 - \int_0^{2\pi} h(r,\sigma)\, d\sigma$$

$$f_\epsilon(r,\sigma) = \frac{h_\epsilon(r,\sigma)}{a(r,\epsilon)}$$

$$g_\epsilon(r,\sigma) = \frac{P(r,\sigma) - h_e(r,\sigma)}{a(r,\epsilon)}$$

It is readily seen that

$$\int_0^{2\pi} g_\epsilon(r,\sigma)\, d\sigma = 1$$

$$0 < f_\epsilon(r,\sigma) \leq \frac{\epsilon}{a(r,\epsilon)} \leq \frac{\epsilon}{1 - 2\pi\epsilon}$$

and

$$a(r,\epsilon) \to 1 \quad \text{as } r \to 1^- \ .$$

Now $f_\epsilon(r,\sigma)$ and $g_\epsilon(r,\sigma)$ are continuous functions of period 2π. Further $f_\epsilon(r,\sigma)$ is of total variation $\frac{2\epsilon}{a(r,\epsilon)}$ over any interval of length 2π.

On the other hand $g_\epsilon(r, \sigma)$ is a needle-like function, vanishing outside of an interval of length $\delta(\epsilon, r)$ about the origin, being otherwise positive valued and enclosing a unit area. It follows that any integral of the form $\int_0^{2\pi} g_\epsilon(r, \sigma)\, w(\sigma)\, d\sigma$ differs from $w(0)$ by less than the oscillation of $w(\sigma)$ in the non-zero interval of $g_\epsilon(r, \sigma)$. It follows from known properties of $P(r, \sigma)$ that $\delta(\epsilon, r)$ approaches zero for each ϵ as $r \to 1^-$.

We write $I(L,r)$ as the sum of two integrals

$$I(L,r) = a(r, \epsilon) \int_{-L}^{L} d\zeta \int_0^{2\pi} d\sigma\, e^{i\beta\zeta} \frac{e^{ik\mathcal{R}}}{\mathcal{R}} f_\epsilon(r, \sigma - \zeta/\alpha)$$

$$(2.17) \qquad + a(r, \epsilon) \int_{-L}^{L} d\zeta \int_0^{2\pi} d\sigma\, e^{i\beta\zeta} \frac{e^{ik\mathcal{R}}}{\mathcal{R}} g_\epsilon(r, \sigma - \zeta/\alpha).$$

Expanding $f_\epsilon(r, \sigma)$ in a Fourier series, we obtain

$$(2.18) \qquad f_\epsilon(r, \sigma) = \sum_{n=-\infty}^{\infty} b_n(r, \epsilon)\, e^{in\sigma}$$

where

$$|b_n(r, \epsilon)| \leq \frac{\mathrm{Var.} f_\epsilon}{n} \leq \frac{2\epsilon}{n\, a(r, \epsilon)} .$$

Substituting this into the first of the integrals of (2.17) and interchanging the order of summation and integration (finite limits) gives

$$a(r, \epsilon) \int_{-L}^{L} d\zeta \int_0^{2\pi} d\sigma\, e^{i\beta\zeta} \frac{e^{ik\mathcal{R}}}{\mathcal{R}} f_\epsilon(r, \sigma - \zeta/\alpha)$$

$$(2.19) \qquad = \sum_{n=-\infty}^{\infty} a(r, \epsilon)\, b_n(r, \epsilon) \int_{-L}^{L} d\zeta \int_0^{2\pi} d\sigma\, e^{i\beta\zeta} \frac{e^{ik\mathcal{R}}}{\mathcal{R}} e^{in(\sigma - \zeta/\alpha)} .$$

Finally if we combine this with the inequalities (2.12) and (2.18) we see that the series (2.19) is majorized by

$$\epsilon \sum \frac{\bar{\bar{c}}}{n^2} \leq \epsilon C_1$$

which converges to zero with ϵ uniformly in L for fixed β and fixed $\rho > a$.

The second integral in 2.17) approaches $\int_{-L}^{L} d\rho \; e^{i\beta\zeta} \; \frac{e^{ikR}}{R}$ as $r \to 1^-$. Further, we note that the oscillation of $e^{i\beta\zeta} \; \frac{e^{ik\mathcal{R}}}{\mathcal{R}}$ in the $\delta(\epsilon, r)$ interval about $\sigma = \zeta/\alpha$ is bounded by

$$\left| \frac{\partial}{\partial\sigma} \left(\frac{e^{ik\mathcal{R}}}{\mathcal{R}} \right) \right| \delta(\epsilon, r) \leq \frac{C_2}{\mathcal{R}^2} \, \delta(\epsilon, r) \; .$$

Hence

$$\left| \int_{-L}^{L} d\zeta \int_{0}^{2\pi} d\sigma \; e^{i\beta\zeta} \; \frac{e^{ik\mathcal{R}}}{\mathcal{R}} \, g_\epsilon\,(r, \, \sigma - \zeta/\alpha) - \int_{-L}^{L} d\zeta \; e^{i\beta\zeta} \; \frac{e^{ikR}}{R} \right|$$

$$\leq C_2 \;\; \delta(\epsilon, r) \int_{-\infty}^{\infty} \frac{d\zeta}{\mathcal{R}^2} \; = \; C_3 \;\; \delta(\epsilon, r)$$

uniformly in L. We have finally that

$$\left| \frac{I(L,r)}{a(r, \epsilon)} - \lim_{r \to 1^-} I(L,r) \right| \leq \frac{C_1 \epsilon}{1 - 2\pi\epsilon} + C_3 \;\; \delta(\epsilon, r) \; .$$

Since ϵ is arbitrary and $\delta(\epsilon, r)$ approaches zero as $r \to 1^-$, it follows that $\lim_{r \to 1^-} I(L,r)$ exists uniformly in L. This concludes the proof of (b). Combining this with (a), it follows from a well known theorem on interchange of limits that

$$I = \lim_{L \to \infty} \; \lim_{r \to 1^-} \; I(L,r) \; = \lim_{r \to 1^-} \; \lim_{L \to \infty} \; I(L,r)$$

$$= \sum_{n=-\infty}^{\infty} 4\pi \; e^{in\varphi} \;\; I_n(\Gamma_n a) \, K_n(\Gamma_n \rho)$$

for $\rho > a$.

It should be remarked that for a point inside of the helix, i.e. $\rho < a$, the vector potential components can be obtained by simply interchanging a and ρ since these two variables enter symmetrically in the integral I.

3. Electromagnetic Field Expressions.

In the previous section it was shown that the integral I converges for $\rho \neq a$. We now define

$$(3.1) \qquad I(\rho, \varphi) = \int_{-\infty}^{\infty} d\zeta \; e^{i\beta\zeta} \; \frac{e^{ikR}}{R}$$

where $R = \sqrt{\zeta^2 + a^2 + \rho^2 - 2a\rho\cos(\varphi - \zeta/\alpha)}$ and β is real. It remains to show that this integral converges for $\rho = a$ provided that $\varphi \neq 0$, that is for a point not on the helix itself. This will be a by-product of our investigation into the derivatives of $I(\rho, \varphi)$ with respect to ρ and φ.

For any pair of values (ρ, φ) different from $(a, 0)$ there is a neighborhood of this point in which R is bounded away from zero. Further R is essentially equal to $|\zeta|$ for large $|\zeta|$. Hence for the points (ρ, φ) of such a neighborhood any integral majorized by M/R^{μ} $(\mu > 1)$ will converge uniformly. This is clearly the case for all derivatives of the integrand of $I(\rho, \varphi)$ with respect to ρ and φ. It follows that

$$(3.2) \qquad \int_{\rho_0}^{\rho} d\rho \int_{-\infty}^{\infty} d\zeta \; \delta_\rho \left(e^{i\beta\zeta} \frac{e^{ikR}}{R} \right) = \int_{-\infty}^{\infty} d\zeta \; e^{i\beta\zeta} \left(\frac{e^{ikR}}{R} - \frac{e^{ikR_0}}{R_0} \right)$$

when R_0 is obtained from R by replacing ρ by ρ_0.

Suppose now that $\rho_0 = a$ where $\rho \neq a$. Then since $I(\rho, \varphi)$ converges, the right-hand side of 3.2) can be written as the sum of two convergent integrals; hence $I(a, \varphi)$ converges for $\varphi \neq 0$. Whether $\rho_0 = a$ or not, the integrand of the outer integral on the left hand side of 3.2) is continuous. Hence

$$\delta_\rho I(\rho, \varphi) = \delta_\rho I(\rho, \varphi) - I(\rho_0, \varphi) = \delta_\rho \left[\int_{\rho_0}^{\rho} d\rho \int_{-\infty}^{\infty} d\zeta \; \delta_\rho \left(e^{i\beta\zeta} \frac{e^{ikR}}{R} \right) \right] = \int_{-\infty}^{\infty} d\zeta \, \delta_\rho \left(e^{i\beta\zeta} \frac{e^{ikR}}{R} \right),$$

where $\delta_\rho[\delta_\varphi]$ designates the partial derivative with respect to ρ $[\varphi]$.

It is clear that a similar argument applies to all derivatives of $I(\rho, \varphi)$ with respect to ρ and φ. Further we now know that I converges for all points that are off the wire. This is clearly independent of the coordinate system used to express the point. It follows therefore by precisely the same argument as above that the derivatives for the components of Π with respect to x, y and z can be obtained by differentiating under the integral sign.

In order to evaluate the derivatives of $I(\rho, \varphi)$ for $\rho \neq a$, we recall the inequality (2.15) which shows that the series expansion for $I(\rho, \varphi)$ [see (2.20)] differentiated termwise with respect to φ, is majorized by $\sum_{-\infty}^{\infty}(\rho/a)^{|n|}$ for $\rho < a$ and by $\sum_{-\infty}^{\infty}(a/\rho)^{|n|}$ for $\rho > a$. It follows that $\partial_\varphi I(\rho, \varphi)$ can be obtained by termwise differentiation of the series expansion for $\rho \neq a$.

A similar result holds for $\partial_\rho I(\rho, \varphi)$. Here we make use of the inequality (2.14) for the case $\rho > a$. Thus

$$\left| K_n'(v) \right| \leq K_n(v) \frac{|n|}{v} \sqrt{1 + \frac{v^2}{n(n-1)}} \quad .$$

For $v \geq \Gamma_n a$, the functions $\frac{|n|}{v}\sqrt{1 + \frac{v^2}{n(n-1)}}$ are uniformly bounded by some positive number M. Hence

$$\left| \Gamma_n I_n(\Gamma_n a) K_n'(\Gamma_n \rho) \right| \leq \Gamma_n M I_n(\Gamma_n a) K_n(\Gamma_n \rho) \leq M_1 (a/\rho)^{|n|} ,$$

so that the series expansion for $I(\rho, \varphi)$ can be differentiated termwise with respect to ρ in case $\rho > a$.

Finally for $\rho < a$, we make use of the corresponding inequality for the I_n function ($n \geq 0$) [see (3.32)]

$$(3.3) \qquad \frac{n}{v^2}\sqrt{1 + \frac{v^2}{n(n+1)}} \leq \frac{1}{v} \frac{I_n'(v)}{I_n(v)} \leq \frac{n}{v^2} \sqrt{1 + \frac{v^2}{n^2}}$$

for $v \leq \Gamma_n a$, the functions $\frac{n}{v}\sqrt{1 + \frac{v^2}{n^2}}$ are uniformly bounded by M'/v for some positive number M'. Hence

$$\left| \Gamma_n I_n'(\Gamma_n \rho) K_n(\Gamma_n a) \right| \leq \frac{M_1'}{\rho} (\rho/a)^{|n|} ,$$

so that again the series expansion can be differentiated termwise with respect to ρ in the case $\rho < a$.

It can now be readily shown that the components of the vector potential (and hence the electromagnetic field components) satisfy the wave equation

$$\Delta \Phi - \epsilon\mu \ \delta_{tt} \Phi = 0$$

at all points (x,y,z) not on the helix. In fact, since we can differentiate under the integral sign this amounts simply to showing that the quantity $e^{-i\omega t} \ \frac{e^{ikr}}{r}$ where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ satisfies the wave equation (Note that $k^2 = \epsilon\mu\omega^2$).

We shall conclude this section by obtaining explicit expressions for the field components from the furmulae (1.2) and (1.6). We first transform the integrals of (1.6) and (1.7). Then we have

$$\begin{aligned} \Pi_\rho &= aF \, e^{i\beta z} \int_{-\infty}^{\infty} d\zeta \ e^{i\beta\zeta} \ \frac{e^{ikr}}{r} \ \sin(\varphi - z/\alpha - \zeta/\alpha) \\ \Pi_\varphi &= aF \, e^{i\beta z} \int_{-\infty}^{\infty} d\zeta \ e^{i\beta\zeta} \ \frac{e^{ikr}}{r} \ \cos(\varphi - z/\alpha - \zeta/\alpha) \\ \Pi_z &= \alpha F \, e^{i\beta z} \int_{-\infty}^{\infty} d\zeta \ e^{i\beta\zeta} \ \frac{e^{ikr}}{r} \end{aligned} \tag{3.4}$$

where

$$r = \sqrt{\zeta^2 + a^2 + \rho^2 - 2a\rho \cos(\varphi - z/\alpha - \zeta/\alpha)} \ .$$

We first compute E. Now

$$\nabla \cdot \Pi = \frac{1}{\rho}\Pi_\rho + \delta_\rho \Pi_\rho + \frac{1}{\rho}\delta_\varphi \Pi_\varphi + \delta_z \Pi_z \ .$$

$$\delta_\rho \Pi_\rho = aF \, e^{i\beta z} \int_{-\infty}^{\infty} d\zeta \ e^{i\beta\zeta} \ \delta_r\left(\frac{e^{ikr}}{r}\right) \frac{\rho - a\cos(\varphi - z/\alpha - \zeta/\alpha)}{r} \sin(\varphi - z/\alpha - \zeta/\alpha)$$

$$\delta_\varphi \Pi_\varphi = aF \, e^{i\beta z} \int_{-\infty}^{\infty} d\zeta \ e^{i\beta\zeta} \left[\delta_r\left(\frac{e^{ikr}}{r}\right) \frac{a\rho \sin(\varphi - z/\alpha - \zeta/\alpha)}{r} \cos(\varphi - z/\alpha - \zeta/\alpha) - \frac{e^{ikr}}{r} \sin(\varphi - z/\alpha - \zeta/\alpha)\right]$$

Combining the terms in $\nabla \cdot \Pi$ and making the appropriate cancellations gives

$$\nabla \cdot \Pi = i\beta \, \Pi_z \, . \tag{3.5}$$

Since $E = \nabla(\nabla \cdot \Pi) + k^2 \Pi$, we have for $\rho > a$

$$E_\rho = i\beta \, \delta_\rho \, \Pi_z + k^2 \Pi_\rho$$

$$= 4\pi i \beta \alpha F \, e^{i\beta z} \sum_{n=-\infty}^{\infty} \Gamma_n I_n(\Gamma_n a) \, K_n'(\Gamma_n \rho) \, e^{in(\varphi - z/\alpha)} \tag{3.6a}$$

$$+ 4\pi \, k^2 a F \, e^{i\beta z} \sum_{n=-\infty}^{\infty} \frac{I_{n-1}(\Gamma_n a) K_{n-1}(\Gamma_n \rho) - I_{n+1}(\Gamma_n a) K_{n+1}(\Gamma_n \rho)}{2i} \, e^{in(\varphi - z/\alpha)} ,$$

$$E_\varphi = \frac{i\beta}{\rho} \delta_\varphi \, \Pi_z + k^2 \Pi_\varphi$$

$$= \frac{4\pi i \beta \alpha F}{\rho} \, e^{i\beta z} \sum_{n=-\infty}^{\infty} in \, I_n(\Gamma_n a) K_n(\Gamma_n \rho) e^{in(\varphi - z/\alpha)} \tag{3.6b}$$

$$+ 4\pi k^2 \, a F \, e^{i\beta z} \sum_{n=-\infty}^{\infty} \frac{I_{n-1}(\Gamma_n a) K_{n-1}(\Gamma_n \rho) + I_{n+1}(\Gamma_n a) K_{n+1}(\Gamma_n \rho)}{2} \, e^{in(\varphi - z/\alpha)} ,$$

$$E_z = i\beta \, \delta_z \, \Pi_z + k^2 \, \Pi_z$$

$$= 4\pi \, (k^2 - \beta^2) \, \alpha F \, e^{i\beta z} \sum_{n=-\infty}^{\infty} I_n(\Gamma_n a) \, K_n(\Gamma_n \rho) \, e^{in(\varphi - z/\alpha)} \tag{3.6c}$$

$$- 4\pi i \beta \, F \, e^{i\beta z} \sum_{n=-\infty}^{\infty} in \, I_n(\Gamma_n a) \, K_n(\Gamma_n \rho) \, e^{in(\varphi - z/\alpha)} \quad .$$

The electric field components for $\rho < a$ are very similar. One can obtain the magnetic field components directly from

$$H = i\omega \epsilon \nabla \times \Pi \quad .$$

Since we shall have no occasion to use these in the present paper, we refrain from writing them out.

4. Determination of the Phase Velocity.

In this section we shall determine the phase velocity β so that the electric field along the helix is 'essentially' zero. Since we assume the helix to be a perfect conductor this is the proper boundary condition for our problem.

The electric field in the helical direction at any point (ρ, φ, z) is

$$(4.1) \qquad E_h = \frac{\alpha}{\sqrt{a^2+\alpha^2}} E_z + \frac{a}{\sqrt{a^2+\alpha^2}} E_\varphi \quad .$$

In order to approach the helix radially, we set $\varphi = z/\alpha$ in 4.1) and try to determine β so that $\lim_{\rho \to a} E_h = 0$. We will suppose that $\rho > a$; the case $\rho < a$ is entirely similar.

Because of the singular nature of the field about an infinitely thin conductor we cannot actually choose β so that $\lim_{\rho \to a} E_h = 0$. Since E becomes infinite as $\rho \to a$, the best we can do is to determine the proper phase velocity β so that

$$(4.2) \qquad \lim_{\rho \to a} \frac{E_h(\beta_0)}{E_h(\beta^2 \neq \beta_0^2)} = 0$$

For $\varphi = z/\alpha$ and $\rho > a$, E_h can be expressed as

$$(4.3) \quad E_h = \frac{4\pi\alpha^2}{\sqrt{a^2+\alpha^2}} (k^2-\beta^2) F e^{i\beta z} \sum_{n=-\infty}^{\infty} I_n(\Gamma_n a) K_n(\Gamma_n \rho)$$

$$+ \frac{4\pi a^2}{\sqrt{a^2+\alpha^2}} k^2 F e^{i\beta z} \sum_{n=-\infty}^{\infty} \frac{I_{n-1}(\Gamma_n a) K_{n-1}(\Gamma_n \rho) + I_{n+1}(\Gamma_n a) K_{n+1}(\Gamma_n \rho)}{2}$$

$$+ 4\pi i \frac{a^2}{\sqrt{a^2+\alpha^2}} \left(\frac{a}{\rho} - 1\right) F e^{i\beta z} \sum_{n=-\infty}^{\infty} in\, I_n(\Gamma_n a) K_n(\Gamma_n \rho) .$$

We shall first prove that the last term in (4.3) approaches zero as $\rho \to a$. In its original integral form this term becomes

$$J(\rho) = \left(\frac{a}{\rho} - 1\right) \frac{1}{a\rho} \partial_\varphi I(\rho, \varphi)_{(\varphi=0)} = \left(\frac{a}{\rho} - 1\right) \int_{-\infty}^{\infty} e^{i\beta\zeta} \partial_R\left(\frac{e^{ikR}}{R}\right) \frac{\sin \zeta}{R} d\zeta$$

aside from a factor independent of ρ. Here

$$R = \sqrt{\zeta^2 + a^2 + \rho^2 - 2a\rho \cos \zeta/\alpha}$$

is an even function of ζ. The $e^{i\beta z}$ can be expanded as

$$e^{i\beta\zeta} = \cos\beta\zeta + i\sin\beta\zeta .$$

Since $\left[\cos\beta\zeta\,\partial_R\left(\frac{e^{ikR}}{R}\right)\frac{\sin\zeta}{R}\right]$ is odd, its integral vanishes. Hence

$$J(\rho) = \left(\frac{a}{\rho} - 1\right)\int_{-\infty}^{\infty}\frac{\sin\zeta\,\sin\zeta\beta}{R}\,\partial_R\left(\frac{e^{ikR}}{R}\right)d\zeta .$$

We note that

$$|\zeta| \le \sqrt{\zeta^2 + (a-\rho)^2} \le R .$$

It follows that there exist constants c_1 and c_2 such that

$$\left|\partial_R\left(\frac{e^{ikR}}{R}\right)\right| \le \frac{c_1}{|\zeta|} , \quad \text{for } |\zeta| \ge 1,$$

$$\le \frac{c_2}{\zeta^2 + (a-\rho)^2} , \text{ for } |\zeta| \le 1 \text{ and } |\rho - a| < 1.$$

Therefore we have

$$J(\rho) \le \left(\frac{a}{\rho} - 1\right)\left[2\int_1^{\infty}\frac{c_1}{2}\,d\zeta + 2\int_0^1\frac{\beta\,c_2\,\zeta^2}{\left[\zeta^2 + (a-\rho)^2\right]^{3/2}}\,d\zeta\right]$$

$$= \left(\frac{a}{\rho} - 1\right)\left[2c_1 + 2\beta c_2\left\{\log\left(\frac{1 + \left[1 + (a-\rho)^2\right]^{1/2}}{\rho - a}\right) - \frac{1}{\left[1 + (a-\rho)^2\right]^{1/2}}\right\}\right]$$

which approaches zero as $\rho \to a^+$.

In determining the limit of E_h as $\rho \to a^+$, we can therefore omit the last term in (4.3). We denote by E_h' the thus abbreviated E_h. We now define the function

$$\Omega(\beta,\rho) = \sum_{n=-\infty}^{\infty} I_n(\Gamma_n a)\,K_n(\Gamma_n\rho) . \tag{4.4}$$

Then for $\varphi = z/\alpha$ and $\rho > a$, we have

$$E_h' = \frac{4\pi F e^{i\beta z}}{\sqrt{a^2 + \alpha^2}}\left[\alpha^2(k^2 - \beta^2)\,\Omega(\beta,\rho) + a^2k^2\,\frac{\Omega(\beta - 1/\alpha,\rho) + \Omega(\beta + 1/\alpha,\rho)}{2}\right] \tag{4.5}$$

It turns out that

$$\lim_{\rho \to a^+} \Omega(\beta, \rho) = \infty \quad , \tag{4.6}$$

so that we can hardly expect to obtain a β for which $\lim_{\rho \to a^+} E_h = 0$. However, we shall also prove that

$$\lim_{\rho \to a^+} \frac{\Omega(\beta, \rho)}{\Omega(\beta_2, \rho)} = 1. \tag{4.7}$$

Hence

$$\lim_{\rho \to a^+} \frac{E_h(\beta)}{E_h(\beta_1)} = \frac{\alpha^2(k^2-\beta^2) + a^2k^2}{\alpha^2(k^2-\beta_1^2) + c^2k^2} \quad :$$

and for $\beta_1^2 \neq \beta^2$, this vanishes if and only if

$$\alpha^2(k^2 - \beta^2) + a^2k^2 = 0$$

In other words the correct propagation constant β_0 is simply

$$\beta_0 = \pm k \frac{\sqrt{a^2 + \alpha^2}}{\alpha} \quad . \tag{4.8}$$

We now prove(4.6). In order to do this we make use of the inequalities(2.14) and(3.3), from which it follows that

$$- \frac{K_n'(v)}{vK_n(v)} - \frac{I_n'(v)}{vI_n(v)} < \frac{n}{v^2}\left[\sqrt{1 + \frac{v^2}{n(n-1)}} - \sqrt{1 + \frac{v^2}{n(n+1)}}\right]$$

The expression on the right is monotonic decreasing achieving a maximum of $\frac{1}{n^2 - 1}$ at $v = 0$. Hence

$$\frac{\left(I_n(v)\, K_n(v)\right)'}{I_n(v)\, K_n(v)} > - \frac{v}{n^2 - 1}$$

and since $I_n(0)\, K_n(0) = \frac{1}{2n}$, we obtain

$$\frac{1}{2n} \geq I_n(v)\, K_n(v) \geq \frac{1}{2n}\, e^{-\frac{v^2}{2(n^2-1)}} \quad .$$

Finally for sufficiently large $|n|$,

$$\Gamma_n = \sqrt{(\beta - n/\alpha)^2 - k^2} \quad > c|n|$$

and hence

$$I_n(\Gamma_n a)\, K_n(\Gamma_n a) \geq \frac{1}{2|n|}\, e^{-c^2} \quad .$$

Since $\Omega(\beta, \rho)$ consists of positive continuous terms, monotonically increasing as $\rho \to a^+$, we have

$$\lim_{\rho \to a^+} \Omega(\beta, \rho) = \Omega(\beta, a) >> \frac{e^{-c^2}}{2} \sum \frac{1}{|n|}$$

which diverges.

We shall next establish (4.7) also by means of the inequalities (2.14) and (3.3). Now clearly

$$\frac{\partial_v \left[I_n(va)\, K_n(v\rho)\right]}{I_n(va) K_n(v\rho)} = a\, \frac{I_n'(va)}{I_n(va)} + \rho \cdot \frac{K_n'(v\rho)}{K_n(v\rho)}$$

$$= v \left\{ a^2\, \frac{I_n'(va)}{va I_n(va)} - \rho^2 \left[\frac{-K_n(v\rho)}{v\rho\, K_n(v\rho)} \right] \right\} .$$

It follows from (2.14) and (3.3) that for $n \geq 0$ and $v \geq 0$

$$\frac{\partial_v \left[I_n(va)\, K_n(v\rho)\right]}{I_n(va) K_n(v\rho)} \leq v \left\{ a^2\, \frac{n}{(va)^2} \sqrt{1 + \frac{(va)^2}{n^2}} - \rho^2\, \frac{n}{(v\rho)^2} \sqrt{1 + \frac{(v\rho)^2}{n^2}} \right\}$$

$$\leq \frac{n}{v} \left\{ \sqrt{1 + \frac{(va)^2}{n^2}} - \sqrt{1 + \frac{(v\rho)^2}{n^2}} \right\} < 0,$$

for $\rho > a$. Likewise ($n \geq 0$ and $v \geq 0$)

$$\frac{\partial_v \left[I_n(va)\, K_n(v\rho)\right]}{I_n(va)\, K_n(v\rho)} \geq v \left\{ \frac{n}{v^2} \sqrt{1 + \frac{(va)^2}{n(n+1)}} - \frac{n}{v^2} \sqrt{1 + \frac{(v\rho)^2}{n(n-1)}} \right\}.$$

It is easily shown that the function within the brackets is negative monotonic increasing for $\rho > a$, achieving a minimum of $\left(- \frac{n(\rho^2 - a^2) + (\rho^2 + a^2)}{n^2 - 1}\right)$ at $v = 0$.

Hence

$$\frac{\partial_v \left[I_n(va)\, K_n(v\rho)\right]}{I_n(va)\, K_n(v\rho)} \geq -\omega_n v$$

when $\omega_n = \frac{n(\rho^2 - a^2) + (\rho^2 + a^2)}{n^2 - 1}$. If we now integrate the above logarithmic derivative from $v > 0$ to $v + \Delta > 0$ we obtain for $n \geq 0$

$$\text{(4.9a)} \quad e^{-\frac{\omega_n}{2}\left[(v + \Delta)^2 - v^2\right]} \leq \frac{I_n\left[(v + \Delta)a\right] K_n\left[(v + \Delta)\rho\right]}{I_n(va)\, K_n(v\rho)} \leq 1 \text{ for } \Delta \geq 0$$

and

$$\text{(4.9b)} \quad 1 \leq \frac{I_n\left[(v + \Delta)a\right] K_n\left[(v + \Delta)\rho\right]}{I_n(va)\, K_n(v\rho)} \leq e^{-\frac{\omega_n}{2}\left[(v + \Delta)^2 - v^2\right]} \quad \text{for } \Delta \leq 0$$

Finally because $I_{-n}(va)\, K_{-n}(v\rho) = I_n(va)\, K_n(v\rho)$, the above inequalities hold for $n \leq 0$ if we replace n in ω_n by $|n|$. We can combine (4.9a) and (4.9b) into

$$\text{(4.10)} \quad e^{-\epsilon(\rho,n)} \leq \frac{I_n\left[(v + \Delta)a\right] K_n\left[(v + \Delta)\rho\right]}{I_n(va)\, K_n(v\rho)} \leq e^{\epsilon(\rho,n)}$$

for all n, $v > 0$, $(v + \Delta) > 0$, and $\rho > a$ where

$$\epsilon(\rho, n) = \frac{|n|(\rho^2 - a^2) + (\rho^2 + a^2)}{n^2 - 1} \left|(v + \Delta)^2 - v^2\right|$$

It will be seen from the expression for $\Omega(\beta, \rho)$ that the appropriate values for v and $(v + \Delta)$ in $I_n(va)\, K_n(v\rho)$ are

$$v_n = \sqrt{(\beta_1 - \tfrac{n}{\alpha})^2 - k^2} = \Gamma_n^{(1)}$$

and

$$(v + \Delta)_n = \sqrt{(\beta_2 - \tfrac{n}{\alpha})^2 - k^2} = \Gamma_n^{(2)} .$$

Hence

$$(v + \Delta)_n^2 - v_n^2 = \beta_2^2 - \beta_1^2 - \frac{2n}{\alpha}(\beta_2 - \beta_1) .$$

It is clear that $\epsilon(\rho,n)$ is the sum of two parts, one of which is of the order of $(\rho-a)$ and the other of the order of $\frac{1}{n}$. Hence given β_1, β_2, and $\eta > 0$, there exists an N and a $\delta > 0$ such that for $|n| > N$ and $0 < \rho - a < \delta$

$$\epsilon(\rho,n) < \eta.$$

The finite sums, $\sum_{-N}^{N} I_n(\Gamma_n^{(1)}a)\, K_n(\Gamma_n^{(1)}\rho)$, remains uniformly bounded as $\rho \to a^+$, whereas the remainder of the sum becomes infinite as $\rho \to a^+$. Hence

$$1 - \frac{\Omega(\beta_1,\rho)}{\Omega(\beta_2,\rho)} = \frac{\sum_{-N}^{N}\{\,\} + \sum_{-\infty}^{\infty}\{\,\} + \sum_{N}^{\infty}\{\,\}}{\Omega(\beta_2,\rho)}$$

$$= \frac{\sum_{-N}^{N}\{\,\}}{\Omega(\beta_2,\rho)} + \frac{\sum_{-\infty}^{-N}\{\,\} + \sum_{N}^{\infty}\{\,\}}{\Omega(\beta_2,\rho)}$$

where $\{\,\}$ stands for the running term

$$\left\{ I_n(\Gamma_n^{(2)}a)\, K_n(\Gamma_n^{(2)}\rho) - I_n(\Gamma_n^{(1)}a)\, K_n(\Gamma_n^{(1)}\rho) \right\};$$

and

$$\left| 1 - \frac{\Omega(\beta_1,\rho)}{\Omega(\beta_2,\rho)} \right| \le \frac{\left| \sum_{-N}^{N}\{\,\} \right|}{\Omega(\beta_2,\rho)} + \frac{(e^{\eta} - e^{-\eta})\left[\Omega(\beta_2,\rho) - \sum_{-N}^{N} I_n(\Gamma_n^{(2)}a) K_n(\Gamma_n^{(2)}\rho) \right]}{\Omega(\beta_2,\rho)}$$

It follows that

$$\overline{\lim_{\rho \to a^+}} \left| 1 - \frac{\Omega(\beta_1,\rho)}{\Omega(\beta_2,\rho)} \right| \le 0 + (e^{\eta} - e^{-\eta}),$$

and since η is arbitrarily small, we have

$$\lim_{\rho \to a^+} \frac{\Omega(\beta_1,\rho)}{\Omega(\beta_2,\rho)} = 1.$$

5. Conclusion.

The net result of our investigation has been to obtain expressions for the electromagnetic field which satisfy the Maxwell equations away from the helix and which satisfy the appropriate boundary conditions along the helix.

The propagation constant in the axial direction was found (4.8) to be

$$\beta_0 = \pm k \frac{\sqrt{a^2 + \alpha^2}}{\alpha} \tag{5.1}$$

which corresponds precisely to the phase velocity one would obtain if the wave traveled along the helix itself with the free space velocity of propagation.

In case

$$(\beta - n/\alpha)^2 > k^2 \tag{5.2}$$

for all n ($n = 0, \pm 1, \pm 2, \ldots$) none of the terms in the expansion of the field equations are unmodified Hankel functions. As a consequence there is no net radial flow of energy. On the other hand for a given helix there is a series of frequency bands for which the inequality (5.2) is not satisfied for all n; in this case a finite number of Hankel function terms do appear in the field expressions. It can be shown (see the appendix) that these terms do cause a net radial flow of energy. One might expect this phenomena to be reflected in the propagation constant β; that is one would expect β to be complex. More precisely, unless there is a source of energy within the wire itself one would expect

$$\operatorname{Im}(\beta) = \frac{1}{2} \frac{\text{Energy radiated per unit distance in z direction}}{\text{Energy flow in the z direction}} \tag{5.3}$$

However, the computed value of (5.1) is consistent with (5.3) for it can be shown (see the appendix) that whereas the energy radiated per unit distance in the z direction is finite, the energy flow in the z direction is infinite. As a matter of fact not only is the total contribution of the set of terms satisfying (5.2) infinite, but the contribution of each of the radiating terms to the energy flow is itself infinite.

In spite of the physical reasonableness of our result, there is, in the case of the radiating helix, reason to be dissatisfied with it. If (5.2) is satisfied for all n, the expansion which we have obtained can be used for helices with conductors of finite thickness. One could, for instance, replace the delta function of section 2 by a finite needle function (say $P(r_0, \sigma)$ with fixed $r_0 < 1$).

The helix would then be constructed from infinitely thin tape. For instance Π_z would be of the type ($\rho > a$)

$$\Pi_z = \alpha F\, e^{i\beta z} \sum_{n=-\infty}^{\infty} a_n\, e^{in(\varphi - z/\alpha)}\, I_n(\Gamma_n a)\, K_n(\Gamma_n \rho)$$

where the (a_n) are Fourier coefficients of the needle function. Repeating the above development one would obtain a real β for the a_n sufficiently close to 1. However in the radiating case, one would not be able to obtain a solution in this way. For no real β would allow the boundary conditions to be satisfied whereas the entire development (starting with the vector potential integrals) is meaningless for complex β. This is also reflected by the fact that a complex β which satisfies the boundary conditions would make the arguments of all of the Hankel functions complex so that the field would decay exponentially. As a consequence we would have no radiating terms which would be contrary to (5.3). Thus in case (5.2) is not satisfied by all n, one would expect a real helical wire to radiate. However, one would not expect the field expressions for finite conductors to have expansions of the type which we have obtained for this case.

APPENDIX

Energy Flow Calculations.

The average rate of flow of energy through an oriented surface S is [see 4, pp. 426-429].

$$W = \int_S P \cdot dS \tag{6.1}$$

where

$$P = \frac{1}{2} \mathrm{Re}\,(E \times H^*). \tag{6.2}$$

We shall compute the radial flow per unit distance in the z-direction as well as the axial flow. In these computations the average Poynting vector P will be expressed in terms of the Bessel function expansions. The expansions for E are given in 3.6); the magnetic field expansions obtained from $H = -i\omega\epsilon \nabla \times \Pi$ are for $\rho > a$:

$$\begin{aligned} H_\rho &= -i\omega\epsilon \left(\frac{1}{\rho}\, \partial_\varphi \Pi_z - \partial_z \Pi_\varphi\right) \\ &= + 4\pi\epsilon\omega \frac{\alpha F}{\rho}\, e^{i\beta z} \sum_{n=-\infty}^{\infty} e^{in(\varphi - z/\alpha)}\, I_n(\Gamma_n a)\, K_n(\Gamma_n \rho) \\ &\quad - 4\pi\epsilon\omega\, aF\, e^{i\beta z} \sum_{n=-\infty}^{\infty} \left(\beta - \frac{n}{\alpha}\right) \frac{I_{n-1}(\Gamma_n a)\, K_{n-1}(\Gamma_n \rho) + I_{n+1}(\Gamma_n a) K_{n+1}(\Gamma_n \rho)}{2}\, e^{in(\varphi - z/\alpha)} \end{aligned} \tag{6.3a}$$

$$H_\varphi = -i\omega\epsilon(\partial_z \Pi_\rho - \partial_\rho \Pi_z)$$

$$(6.3b) \qquad = +4\pi\epsilon\omega i\alpha F\, e^{i\beta z}\sum_{n=-\infty}^{\infty}\Gamma_n I_n(\Gamma_n a)\, K_n'(\Gamma_n\rho)\, e^{in(\varphi - z/\alpha)}$$

$$- 4\pi\epsilon\omega i\, a\, F\, e^{i\beta z}\sum_{n=-\infty}^{\infty}(\beta - n/\alpha)\,\frac{I_n(\Gamma_n a)\, K_{n-1}(\Gamma_n\rho) - I_{n+1}(\Gamma_n a)\, K_{n+1}(\Gamma_n\rho)}{2}$$

$$H_z = i\omega\epsilon\left(\frac{1}{\rho}\,\partial_\rho(\rho\,\Pi_\varphi) - \frac{1}{\rho}\,\partial_\varphi \Pi_\rho\right)$$

$$= -4\pi\epsilon\omega i\,\frac{aF}{\rho}\, e^{i\beta z}\sum_{n=-\infty}^{\infty}\frac{I_{n-1}(\Gamma_n a)\, K_{n-1}(\Gamma_n\rho) + I_{n+1}(\Gamma_n a) K_{n+1}(\Gamma_n\rho)}{2}\, e^{in(\varphi - z/\alpha)}$$

$$(6.3c) \qquad - 4\pi\epsilon\omega i\, aF\, e^{i\beta z}\sum_{n=-\infty}^{\infty}\Gamma_n\frac{I_{n-1}(\Gamma_n a) K'_{n-1}(\Gamma_n\rho) + I_{n+1}(\Gamma_n a) K'_{n+1}(\Gamma_n\rho)}{2}\, e^{in(\varphi - z/\alpha)}$$

$$+ 4\pi\epsilon\omega i\,\frac{aF}{\rho}\, e^{i\beta z}\sum_{n=-\infty}^{\infty}\Gamma_n\frac{I_n(\Gamma_n a)\, K_{n-1}(\Gamma_n\rho) - I_{n+1}(\Gamma_n a)\, K_{n+1}(\Gamma_n\rho)}{2}\, e^{in(\varphi - z/\alpha)}.$$

a. Average radial flow per unit distance in the z-direction.

In order to compute the average radial flow of energy per unit distance in the z direction, we choose for S the surface of a cylinder $\rho = R$ for $0 \leq z \leq 1$ and take the limit as $R \to \infty$. The component of P normal to S is $P = 1/2\, \mathrm{Re}(E_\varphi H_z^* - E_z H_\varphi^*)$. Since the n^{th} terms of the series expansions for E and H are majorized by $(\frac{a}{R})^{|n|}$ the series will be uniformly convergent. Hence the integral may be obtained by termwise integration. The integration with respect to φ will cause all cross-terms to vanish, hence we need only consider integrals of products with like exponents in φ. Further since we are only interested in the limit as $R \to \infty$ we can neglect all terms in P_ρ of order $\frac{1}{\rho^\mu}$ where $\mu > 1$. Since all terms in E and H are at of the order $\frac{1}{\rho^{1/2}}$, we can neglect all terms which decay exponentially - that is terms for which $(\beta - n/\alpha)^2 > k^2$ - and all terms in E and H which are of order $\frac{1}{\rho^\mu}$ where $\mu > \frac{1}{2}$. There remains only the finite number of terms for which $(\beta - n/\alpha)^2 < k^2$ and for these the significant terms in E and H can be reduced to the form

$$E_\varphi \sim 2\pi^2 i F e^{i\beta z} ak^2 J_n'(\gamma_n a) H_n'(\gamma_n \rho) e^{in(\varphi - z/\alpha)}$$

$$E_z \sim 2\pi^2 i F e^{i\beta z} \left[\alpha(k^2-\beta^2)+\beta_n\right] J_n(\gamma_n a) H_n(\gamma_n \rho) e^{in(\varphi - z/\alpha)}$$

$$H_\varphi \sim -2\pi^2 \epsilon\omega F e^{i\beta z} \left[\alpha\gamma_n - (\beta - \tfrac{n}{\alpha})\right] J_n(\gamma_n a) H_n'(\gamma_n \rho) e^{in(\varphi - z/\alpha)}$$

$$H_z \sim -2\pi^2 \epsilon\omega F e^{i\beta z} \, a\gamma_n J_n'(\gamma_n a) H_n(\gamma_n \rho) e^{in(\varphi - z/\alpha)} .$$

The corresponding term in

$$W_\rho(R) = R\int_0^1 ds \int_0^{2\pi} d\varphi \, P_\rho$$

is simply

$$\frac{R}{2} \mathcal{R}e \left[8\pi^5 \frac{\epsilon\omega}{\gamma_n} |F|^2 i - \left\{ \epsilon k^2 \gamma_n^2 \left[J_n'(\gamma_n a)\right]^2 H_n'(\gamma_n R) H_n^*(\gamma_n R) + \left[\alpha(k^2-\beta^2)+\beta_n{}^2\right] \left[J_n(\gamma_n a)\right]^2 H_n(\gamma_n R) H_n'^*(\gamma_n \beta) \right\} \right]$$

Finally

$$H_n H_n'^* = (J_n + iN_n)(J_n' - iN_n') = (J_n J_n' + N_n N_n') + i(J_n' N_n - J_n N_n')$$

$$= \frac{-2i}{\pi\gamma_n R} .$$

Hence

$$W_\rho(\infty) = 8\pi^4 \epsilon\omega |F|^2 \sum_{(\beta - n/\alpha)^2 < K^2} \left\{ a^2k^2 \left[J_n'(\gamma_n a)\right]^2 + \frac{\left[\alpha(k^2-\beta^2)+\beta_n\right]^2}{\gamma_n^2} \left[J_n(\gamma_n a)\right]^2 \right\}$$

In the radiating case, therefore, W_ρ is finite and positive.

b). Average axial flow of energy.

In order to compute the axial rate of flow of energy we choose for S a surface in the z = 0 plane consisting of the interior of a circle of radius $(a - \epsilon)$ plus the interior of the ring $(a + \epsilon) \leq \rho \leq R$ and then take the limit as $\epsilon \to 0$ and

and $R \to \infty$. The component of P normal to S is $P_z = \frac{1}{2} \; R_e(E_\rho H_\varphi^* - E_\varphi H_\rho^*)$. Again we need only consider product terms containing like exponents in φ. We shall consider the radiating and the non-radiating terms separately. It will turn out that both lead to infinite flows; however the sum of the integrals of the non-radiating terms become singular as $\epsilon \to 0$ whereas the integrals of the radiating terms become singular as $R \to \infty$.

In the case of $e^{in\varphi}$ terms for which $(\beta - n/\alpha)^2 > k^2$, we shall group its sub-terms in powers of $\frac{1}{n}$. All terms of order $\frac{1}{n^\mu}$ for $\mu > 1$ will converge to a finite sum; hence we may disregard all terms not of order $\frac{1}{n}$. We will find that the terms of order $\frac{1}{n}$ are all of the same sign and hence the limit of the integral as $\epsilon \to 0$ will be equal to the sum of the termwise integrals.

We note that any integral of the type

$$K_n(\Gamma_n^2 \rho) \int_0^a \rho I_n(\Gamma_n \rho)^2 \, d\rho \quad + \quad I_n(\Gamma_n a)^2 \int_a^\infty \rho K_n(\Gamma_n \rho)^2 \, d\rho$$

is of the order of $\frac{1}{n^2}$. This applies equally well if some of the subscripts are $(n \pm 1)$ or if some of these terms are derivatives. Hence only those terms in E and H which have factors of n or Γ_n will lead to terms of order $\frac{1}{n}$ in P_z. We write the significant part of these terms below for $\rho > a$ $(z=0)$

$$E_\rho \sim 4\pi \, i \, F \beta \alpha \, \Gamma_n I_n(\Gamma_n a) \, K_n'(\Gamma_n \rho) \; e^{in\varphi}$$

$$E_\varphi \sim -4\pi \, F \frac{\beta \alpha n}{\rho} I_n(\Gamma_n a) \, K_n(\Gamma_n \rho) \; e^{in\varphi}$$

$$H_\rho \sim 4\pi \, F \omega \epsilon \left\{ \frac{\alpha n}{\rho} I_n(\Gamma_n a) K_n(\Gamma_n \rho) + \frac{an}{2\alpha} \left[I_{n-1}(\Gamma_n a) K_{n-1}(\Gamma_n \rho) + I_{n+1}(\Gamma_n a) K_{n+1}(\Gamma_n \rho) \right] \right\} e^{in\varphi}$$

$$H_\varphi \sim 4\pi \, i \, F \omega \epsilon \left\{ \alpha \Gamma_n I_n(\Gamma_n a) K_n'(\Gamma_n \rho) - \frac{an}{2\alpha} \left[I_{n-1}(\Gamma_n a) K_{n-1}(\Gamma_n \rho) - I_{n+1}(\Gamma_n a) K_{n+1}(\Gamma_n \rho) \right] \right\} \; e^{in\varphi}$$

Thus

$$P_z \sim 8\pi^2 |F|^2 \omega \epsilon \rho \left\{ \alpha^2 \Gamma_n^2 \left[I_n(\Gamma_n a) \, K_n'(\Gamma_n \rho) \right]^2 + \alpha^2 \, \frac{n^2}{\rho^2} \left[I_n(\Gamma_n a) \, K_n(\Gamma_n \rho) \right]^2 \right.$$

$$+ \frac{a}{2} \; n \Gamma_n \left[J_{n-1}(\Gamma_n a) \, K_{n-1}(\Gamma_n \rho) - I_{n+1}(\Gamma_n a) K_{n+1}(\Gamma_n \rho) \right] I_n(\Gamma_n a) \, K_n'(\Gamma_n \rho)$$

$$\left. + \frac{a}{2} \; \frac{n^2}{P} \left[I_{n-1}(\Gamma_n a) \, K_{n-1}(\Gamma_n \rho) + I_{n+1}(\Gamma_n a) K_{n+1}(\Gamma_n \rho) \right] I_n(\Gamma_n a) \, K_n(\Gamma_n \rho) \right\}$$

Now it is clear that the first, second, and fourth terms within the bracket are all of the same sign. It is further readily shown that the integrals of these terms are actually bounded from below by $\left|\frac{c}{n}\right|$ for $|n|$ sufficiently large. The third term is the only troublesome term since it is of a different sign. However, we shall show that its integral is actually of order $\frac{1}{n^2}$. For $\rho > a$

$$(6.4)\quad I_{n-1}(\Gamma_n a)\,K_{n-1}(\Gamma_n \rho) - J_{n+1}(\Gamma_n a)\,K_{n+1}(\Gamma_n \rho) = \frac{2n}{a\Gamma_n}\,I_n(\Gamma_n a)\,K_n'(\Gamma_n \rho) + \frac{2n}{\rho\Gamma_n}\,I_n'(\Gamma_n a)\,K_n(\Gamma_n \rho)$$

Making use of the inequalities 2.14) and 3.3), we get

$$\left|\frac{I_n(\Gamma_n a)\,K_n'(\Gamma_n \rho)}{a} + I_n'\,\frac{(\Gamma_n a)\,K_n'(\Gamma_n \rho)}{\rho}\right| < \frac{I_n(\Gamma_n a)\,K_n(\Gamma_n \rho)}{a\rho}\,\frac{n}{\Gamma_n} \times \left(\sqrt{1+\frac{\Gamma_n^2\rho^2}{n(n-1)}} - \sqrt{1+\frac{\Gamma_n^2 a^2}{n(n+1)}}\right).$$

Further

$$\sqrt{1+\frac{\Gamma_n^2\rho^2}{n(n-1)}} - \sqrt{1+\frac{\Gamma_n^2 a^2}{n(n+1)}} < \frac{\Gamma_n \rho}{n(n-1)} - A_n$$

where

$$\frac{\Gamma_n a}{\sqrt{n(n-1)}} - A_n = \sqrt{1+\frac{\Gamma_n^2 a^2}{n(n-1)}} - \sqrt{1-\frac{\Gamma_n^2 a^2}{n(n+1)}} < \frac{c}{|n|}.$$

Using the right hand side as a dominating function it is readily shown that the integral of ρ times the quantity 6.4) from a to ∞ is of the order of $\frac{1}{n^2}$. A similar analysis applies for $\rho < a$. It follows that aside from a finite part W consists of an infinite number of terms, each bounded from below by $\left|\frac{c}{n}\right|$ (for $\beta > 0$) and hence the non-radiating terms contribute an infinite flow in the $[\text{sign}(\beta)]$ z-direction.

Finally we compute the contribution due to the finite set of radiating terms in the expansion. Here we consider only the highest order terms in ρ.

For $\rho > a$

$$E_\rho \sim 2\pi^2 F \left(\frac{nk^2}{\gamma_n} - \alpha\beta\gamma_n \right) J_n(\gamma_n a)\, H'_n(\gamma_n \rho)\, e^{in\varphi}$$

$$E_\varphi \sim 2\pi^2\, iF\, a\, k^2\, J'_n(\gamma_n a)\, H'_n(\gamma_n \rho)\, e^{in\varphi}$$

$$H_\rho \sim -2\pi^2 \omega \epsilon F\, a \left(\beta - \frac{n}{\alpha}\right) J'_n(\gamma_n a)\, H'_n(\gamma_n \rho)\, e^{in\varphi}$$

$$H_\varphi \sim 2\pi^2 \omega \epsilon iF \left[-\frac{n}{\gamma_n}\left(\beta - \frac{n}{\alpha}\right) + \alpha\gamma_n \cdot J_n(\gamma_n a)\, H'_n(\gamma_n \rho)\, e^{in\varphi} \right].$$

Thus

$$P_z \sim 2\pi^4 \omega \epsilon |F|^2 \left[\frac{\left[\alpha(k^2 - \beta^2) + \beta_n\right]^2}{\gamma_n^2} \cdot J_n^2(\gamma_n a) + J'_n(\gamma_n a)^2 \right]$$

$$\cdot \left[\left(\beta - \frac{n}{\alpha}\right) \left| H'_n(\gamma_n \rho) \right|^2 \right] .$$

Finally

$$\left| H'_n(\gamma_n \rho) \right|^2 \sim \frac{1}{\gamma_n \rho} .$$

It follows that each radiating term contributes to the axial energy flow an amount

$$\int_0^{2\pi} d\varphi \int_0^{\infty} d\rho\, \rho\, P_z \sim \left(\beta - \frac{n}{\alpha}\right) \cdot \infty .$$

This concludes the discussion for the axial flow of energy.

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